On the mixing of axi-symmetric ducted jets

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Because of practical application to jet pumps, ejectors, furnaces and similar devices, the turbulent discharge of a round jet into a coaxial duct and the mixing patterns in the various regions into which the flow may be divided, are of considerable interest. In this paper the mixing of an incompressible jet with a similar fluid in a cylindrical tube is considered up to the plane which marks the disappearance of potential flow. Under the assumption of similarity of velocity profile and with neglect of the wall boundary layer and nozzle wake, the continuity and momentum equations, in integral form, are solved for the velocities and mixing region radii at any given section. Prandtl's momentum transfer hypothesis may be used to determine the dependence of these on distance downstream. By examining the various flow regimes in detail this analysis is formally able to cover ratios of primary to secondary flow velocities of from one to infinity and, similarly, all ratios of duct to nozzle diameters, thereby extending earlier investigations. It also corrects work on similar basis in which inappropriate linearisations were made. The 'exact' results constitute a basis from which extension to include additional effects may be made.

Keywords: *turbulent flow, axi-symmetric mixing, ducted jets*

The use of ducted jets to provide jet pumps and ejector systems is an established, standard engineering technique. As early as 1939 a bibliography on the subject was assembled by Flugel[1], and in 1972 the British Hydromechanics Research Association (BHRA) produced a 'State of the Art Review and Bibliography'. A second (revised) edition by Bonnington, King and Hemmings[2] was published in 1976.

Understandably, to quote a typical paper, the general analytic approach has been that 'in which empirical duct loss factors are applied to the convenient theoretical framework of one-dimensional analysis'. Nevertheless, attempts at more sophisticated analyses have been successful, and especial mention must be made of the work of Craya and Curtet[3], with extensions by Curtet[4] and Curtet and Ricou[5], the work of Hill[6], and of Razinsky and Brighton[7]. These authors all considered an incompressible jet, discharging from an opening of finite size into a duct, followed by turbulent mixing between the 'primary' jet flow and the peripheral, slower, 'secondary' flow of the same fluid. Usually the analyses dealt with axi-symmetric motion in a duct of constant cross-section. These authors were in essential agreement in using the continuity equation, expressed as an integral over the cross-section of the duct, and the momentum integral equation, applied across the mixing layer, and in taking as a further relation an alternative formulation of an integral of the weighted boundary-layer equations in which the turbulent shearing stress appeared either explicitly at a point or implicitly in integrated form. In the mixing region an analytic expression for the velocity profile was assumed and outside this region the flow was considered to be potential. In addition Razinsky and Brighton[7] attempted to allow for the boundary layer on the inner wall of the duct, although Reid[8] and others have shown experimentally that the boundary layer is generally relatively thin in ducts of modest length, and for many engineering purposes its presence may be neglected.

Hill noted[6] that any alternative integrated form of the boundary layer equation could be employed if the appropriate details of the shearing stress were known, and he argued further that his use of the moment-of-momentum integral equation was preferable to the method adopted by Squire and Trouncer[9] in their analysis of an (unducted) jet in streaming flow because it obviates choice of an arbitrary limit for the second application of the momentum integral equation together with the use of a phenomenological theory to express the turbulent shearing stress at that point in terms of 'known' quantities. However, it is the work of Mikhail[10], which is closely modelled on the free jet analysis of Squire and Trouncer, to which most reference has subsequently been made. Unfortunately, the approximations made by Mikhail when linearizing the equations are not self-consistent and the results obtained, though appearing plausible, are erroneous.

The purpose here is to correct and extend the analysis by Mikhail, simultaneously extending an earlier analysis by Middleton[11]. Concern is with the basic situation with a round turbulent incompressible jet discharging into a cylindrical coaxial duct

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containing the same fluid. As a one-dimensional analysis provides a basis for subsequent enhancement, the 'exact' solutions which the present model affords can be used similarly as a framework for the investigation of more complicated flows wherein additional effects become important.

The essential details of the flow are shown in Fig. 1. Under the assumptions of no boundary layer on the inner wall of the duct and no wake from the lip of the nozzle, there are two basic configurations, A and B, each with subregions, to consider. The type of flow depends on whether the mixing region reaches the wall of the duct before it reaches the axis and the main parameter governing the situation is the ratio of the duct diameter to that of the jet at the discharge plane. Roman numerals for the Regions are used following, in part, the scheme adopted by Razinsky and Brighton.

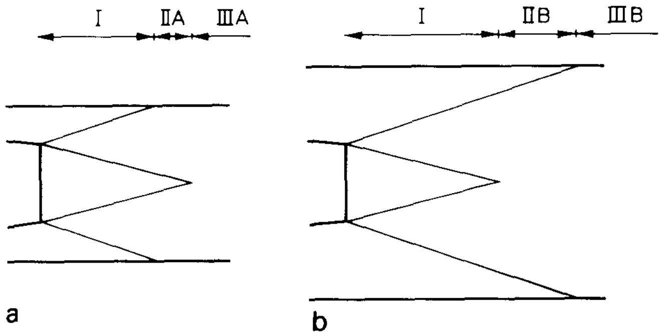

Fig 1 Schematic flow configurations and Regions (a) Type A (b) Type B. (The generators will not, in general, be straight)

Whether the flow be of Type A or Type B there will be no essential difference in the analysis for Region I for which there is potential flow in both primary and secondary streams. However, analysis in Region II will depend on the identity of the single potential flow and the cases must be treated separately. The flow in Region III can be expected to exhibit both jet- and pipe-flow characteristics. Wall effects will generally be greater and it is possible to get large recirculatory zones which may even extend upstream into Region II. No attempt will be made to cover these more complex situations.

Flow in the Regions

Region I

For the potential flows in this region:

$$U_1 \frac{dU_1}{dx} = -\frac{1}{\rho}\frac{dP}{dx} = U_2 \frac{dU_2}{dx} \tag{1}$$

where $U_1(x)$ is the velocity in the primary flow and $U_2(x)$ the velocity in the secondary flow at a distance x downstream from the discharge plane of the jet. The pressure across a section will be constant. Lengths are non-dimensionalised by taking the primary jet to be of unit radius at $x=0$. Discussion of the non-dimensionalisation of quantities may be found in the Appendix.

Notation

A	Duct and jet flow configuration, (Fig 1a)
B	Duct and jet flow configuration, (Fig 1b)
U_1	Potential flow velocity in the jet core, (Regions I, IIA), (dimensionless after Eq 4)
U_{1^*}	Dimensionless primary potential flow velocity
U_1^*	Dimensionless non-potential axial velocity, (Region IIB)
U_2	Potential flow velocity in the annular regime, (Regions I, IIB), (dimensionless after Eq 4)
U_{2^*}	Dimensionless secondary potential flow velocity
U_2^*	Dimensionless non-potential peripheral velocity, (Region IIA)
U_p	Primary jet velocity at efflux plane $x=0$
U_{p^*}	Primary jet velocity non-dimensionalised to unit value
U_s	Secondary annular velocity at the efflux plane
U_{s^*}	Secondary annular velocity non-dimensionalised to value V
V	Ratio of secondary to primary velocities at the efflux plane, (U_s/U_p)
W	Ratio of secondary to primary potential velocities, (U_2/U_1)
a	Ratio of the duct radius to that of the jet at discharge plane $x=0$
c, c_I, c_{IIA}, c_{IIB}	Empirical 'mixing length' constants appropriate to the Regions
$f(s)$	Velocity profile function
p	Static pressure
r	Radial distance
r_1	Inner radius of the mixing region, (Regions I, IIA)
r_2	Outer radius of the mixing region, (Regions I, IIB)
u	Velocity component in the axial direction
x	Distance downstream, measured from the discharge plane of the jet
ρ	Fluid density, (constant)
$\tau(r)$	Turbulent shearing stress
$'$	Denotes differentiation with respect to the argument

Integrating and using the values at the efflux plane:

$$U_1^2 - U_2^2 = U_p^2 - U_s^2 \tag{2}$$

If

$$U_{p^*} = \frac{U_p}{U_p} = 1, \qquad V \equiv U_{s^*} = \frac{U_s}{U_p} \tag{3a, b}$$

$$U_{1^*} = \frac{U_1}{U_p}, \qquad U_{2^*} = \frac{U_2}{U_p} \tag{3c, d}$$

let

$$W = \frac{U_2}{U_1} \tag{4}$$

and then remove the asterisks, the dimensionless potential velocities become:

$$U_1 = \left(\frac{1-V^2}{1-W^2}\right)^{1/2}, \qquad U_2 = W\left(\frac{1-V^2}{1-W^2}\right)^{1/2} \tag{5a, b}$$

In the mixing region it is assumed that the velocity profile may be written:

$$u = U_1 + (U_1 - U_2) f\left(\frac{r-r_1}{r_2-r_1}\right) \qquad r_1 \leq r \leq r_2 \tag{6}$$

From the equation of continuity:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^a 2\pi\rho r u \,\mathrm{d}r = 0 \tag{7}$$

under the stated assumptions after integration and use of the values at $x=0$:

$$r_1^2 U_1 + (a^2 - r_2^2) U_2 + 2\int_{r_1}^{r_2} ru \,\mathrm{d}r = 1 + (a^2-1)V \tag{8}$$

Equating of the rate of change of momentum and the applied forces for the section which extends from the axis to the outer edge of the mixing region gives:

$$U_2 \frac{\partial}{\partial x}\int_0^{r_2} \rho r u \,\mathrm{d}r - \frac{\partial}{\partial x}\int_0^{r_2} \rho r u^2 \,\mathrm{d}r - \frac{1}{2} r_2^2 \frac{\mathrm{d}p}{\mathrm{d}x} = 0 \tag{9}$$

Substitution of Eq (1) followed by integration and the use of initial values then yields:

$$r_1^2 U_1^2 + (\tfrac{1}{2}a^2 - r_2^2) U_2^2 + 2\int_{r_1}^{r_2} ru^2 \,\mathrm{d}r = 1 + (\tfrac{1}{2}a^2 - 1)V^2 \tag{10}$$

When the velocity profile (Eq (6)) is substituted into Eqs (8) and (10) they become:

$$\alpha_{11} r_1^2 + \alpha_{21} r_1 r_2 + \alpha_{31} r_2^2 + \alpha_{41} = 0 \tag{11}$$

$$\alpha_{12} r_1^2 + \alpha_{22} r_1 r_2 + \alpha_{32} r_2^2 + \alpha_{42} = 0 \tag{12}$$

where

$$\alpha_{11} = \beta_{10}(U_1 - U_2) \tag{13a}$$

$$\alpha_{21} = \beta_{20}(U_1 - U_2) \tag{13b}$$

$$\alpha_{31} = \beta_{30}(U_1 - U_2) \tag{13c}$$

$$\alpha_{41} = -\{1 - V + a^2(V - U_2)\} \tag{13d}$$

$$\alpha_{12} = (\beta_{11}U_1 + \beta_{12}U_2)(U_1 - U_2) \tag{14a}$$

$$\alpha_{22} = (\beta_{21}U_1 + \beta_{22}U_2)(U_1 - U_2) \tag{14b}$$

$$\alpha_{32} = (\beta_{31}U_1 + \beta_{32}U_2)(U_1 - U_2) \tag{14c}$$

$$\alpha_{42} = -\{1 - V^2 + \tfrac{1}{2}a^2(V^2 - U_2^2)\} \tag{14d}$$

$$\beta_{10} = 1 - 2\gamma_{01} + 2\gamma_{11} \tag{15a}$$

$$\beta_{11} = 1 - 2\gamma_{02} + 2\gamma_{12} \tag{15b}$$

$$\beta_{12} = 2\beta_{10} - \beta_{11} = 1 - 4\gamma_{01} + 4\gamma_{11} + 2\gamma_{02} - 2\gamma_{12} \tag{15c}$$

$$\beta_{20} = 2\gamma_{01} - 4\gamma_{11} \tag{15d}$$

$$\beta_{21} = 2\gamma_{02} - 4\gamma_{12} \tag{15e}$$

$$\beta_{22} = 2\beta_{20} - \beta_{21} = 4\gamma_{01} - 8\gamma_{11} - 2\gamma_{02} + 4\gamma_{12} \tag{15f}$$

$$\beta_{30} = 1 - \beta_{10} - \beta_{20} = 2\gamma_{11} \tag{15g}$$

$$\beta_{31} = 1 - \beta_{11} - \beta_{21} = 2\gamma_{12} \tag{15h}$$

$$\beta_{32} = 1 - \beta_{12} - \beta_{22} = 2\beta_{30} - \beta_{31} = 4\gamma_{11} - 2\gamma_{12} \tag{15i}$$

and

$$\gamma_{mn} = \int_0^1 s^m f^n(s)\,\mathrm{d}s \quad \text{where } s = \frac{r - r_1}{r_2 - r_1} \tag{16a, b}$$

The choice of the velocity profile according to Eq (6) implies that the $\{\gamma_{mn}\}$ are constants, and their values for $m=0, 1$ and $n=1, 2$; those of the nine linear combinations listed, and five further $\{\beta_{ij}\}$ which are constants arising from subsequent analysis, are given in Table 1 for the specific profile:

$$f(s) = \tfrac{1}{2}(1 + \cos \pi s) \tag{17}$$

The velocity profile is considered in more detail later.

Table 1 Values of the $\{\gamma_{mn}\}$ and some of the $\{\beta_{mn}\}$ for the trigonometric profile $f(s) = \frac{1}{2}(1+\cos \pi s)$

γ_{01}	γ_{02}	γ_{11}	γ_{12}		
$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}-\frac{1}{\pi^2}$	$\frac{3}{16}-\frac{1}{\pi^2}$		
β_{10}	β_{11}	β_{12}	β_{20}	β_{21}	β_{22}
$\frac{1}{2}-\frac{2}{\pi^2}$	$\frac{5}{8}-\frac{2}{\pi^2}$	$\frac{3}{8}-\frac{2}{\pi^2}$	$\frac{4}{\pi^2}$	$\frac{4}{\pi^2}$	$\frac{4}{\pi^2}$
β_{30}	β_{31}	β_{32}	β_{33}	β_{34}	
$\frac{1}{2}-\frac{2}{\pi^2}$	$\frac{3}{8}-\frac{2}{\pi^2}$	$\frac{5}{8}-\frac{2}{\pi^2}$	$\frac{1}{4}$	$\frac{1}{8}-\frac{2}{\pi^2}$	
β_{03}	β_{13}			β_{43}	
$\frac{1}{2}+\frac{2}{\pi^2}$	$\frac{3}{8}+\frac{2}{\pi^2}$			$\frac{1}{8}+\frac{2}{\pi^2}$	

Eqs (11) and (12) can be manipulated algebraically to produce the equivalent pair:

$$\beta_{11} r_1^2 + \beta_{21} r_1 r_2 + \beta_{31} r_2^2 + \beta_{41} = 0 \tag{18}$$

$$\beta_{12} r_1^2 + \beta_{22} r_1 r_2 + \beta_{32} r_2^2 + \beta_{42} = 0 \tag{19}$$

where

$$\beta_{41} = (\alpha_{42} - 2\alpha_{41}U_2)/(U_1 - U_2)^2, \tag{20a}$$

$$\beta_{42} = -(\alpha_{42} - 2\alpha_{41}U_1)/(U_1 - U_2)^2 \tag{20b}$$

a worthwhile simplification, with particular application later in the variation with distance downstream, because in this formulation the coefficients of the radii are no longer functions of U_1, U_2.

If

$$\delta_{mn}=\beta_{m1}\beta_{n2}-\beta_{m2}\beta_{n1}=-\delta_{nm} \tag{21}$$

the elimination of r_1 between Eqs (18) and (19) produces a quartic equation for r_2 which has the form of a quadratic equation for r_2^2:

$$(\delta_{13}^2+\delta_{12}\delta_{32})r_2^4+(2\delta_{13}\delta_{14}+\delta_{21}\delta_{24})r_2^2+\delta_{14}^2=0 \tag{22}$$

As $a\geqslant r_2\geqslant 1\geqslant r_1\geqslant 0$, the appropriate solution is found to be:

$$r_2=\left[-\left\{\frac{\frac{1}{2}\delta_{21}\delta_{24}+\delta_{13}\delta_{14}+|\delta_{21}|(\frac{1}{4}\delta_{24}^2-\delta_{14}\delta_{34})^{1/2}}{\delta_{13}^2+\delta_{21}\delta_{23}}\right\}\right]^{1/2} \tag{23}$$

and, correspondingly

$$r_1=\left[-\left\{\frac{\frac{1}{2}\delta_{23}\delta_{24}+\delta_{31}\delta_{34}+|\delta_{23}|(\frac{1}{4}\delta_{24}^2-\delta_{34}\delta_{14})^{1/2}}{\delta_{31}^2+\delta_{23}\delta_{21}}\right\}\right]^{1/2} \tag{24}$$

Once the $\{\beta_{ij}\}$ are known for the section in question the computations of r_1 and r_2 are straightforward.

Region IIA

In the A-configuration Region II possesses potential flow only in the core, where $u=U_1$. Outside this central part the velocity is expressed as:

$$u=U_2^*+(U_1-U_2^*)f\left(\frac{r-r_1}{a-r_1}\right) \qquad r_1\leqslant r\leqslant a \tag{25}$$

so that for any given section in this Region it is necessary to determine r_1 and the non-potential velocity U_2^*.

It is now found that the continuity and momentum integral equation may be simplified in a manner similar to the previous case considered to give:

$$\beta_{11}r_1^2+\beta_{21}ar_1+\beta_{31}a^2+\beta_{41}^*=0 \tag{26}$$

$$\beta_{12}r_2^2+\beta_{22}ar_2+\beta_{32}a^2+\beta_{42}^*=0 \tag{27}$$

where

$$\beta_{41}^*=(\alpha_{42}^*-2\alpha_{41}^*U_2^*)/(U_1-U_2^*)^2, \tag{28a}$$

$$\beta_{42}^*=-(\alpha_{42}^*-2\alpha_{41}^*U_1)/(U_1-U_2^*)^2 \tag{28b}$$

and

$$\alpha_{41}^*=-\{1-V+a^2(V-U_2^*)\}, \tag{29a}$$

$$\alpha_{42}^*=-[1-V^2+\tfrac{1}{2}a^2\{(V^2-U_2^{*2})-(1-U_1^2)\}] \tag{29b}$$

Elimination of U_2^* between Eqs (26) and (27) yields a quartic equation in r_1, one root of which is unity. Therefore, it is necessary to obtain the positive root of the cubic equation:

$$\beta_{10}^2r_1^3+\beta_{10}(\beta_{10}+2\beta_{20})ar_1^2-(\beta_{10}^2-\beta_{20}^2+\beta_{12}\beta_{40})a^2r_1 -\{(\beta_{10}+\beta_{20})^2+(1-\beta_{32})\beta_{40}\}a^3=0 \tag{30}$$

where

$$\beta_{40}=-\{1-V+a^2(V-U_1)\}^2/[a^2\{(1-\tfrac{1}{2}a^2)(1-V^2) +\tfrac{1}{2}a^2(V^2-U_1^2)-2U_1\{1-V+a^2(V-U_1)\}\}] \tag{31}$$

Having determined r_1, U_2^* is then given by:

$$U_2^*=U_1+\left\{\frac{1-V+a^2(V-U_1)}{a^2-(\beta_{10}r_1^2+\beta_{20}ar_1+\beta_{30}a^2)}\right\} \tag{32}$$

Region IIB

For Region II of the B-configuration there is only potential flow in the annular region, and in which $u=U_2$. Accordingly for the mixing region which now extends to the centre of the duct:

$$u=U_2+(U_1^*-U_2)f\left(\frac{r}{r_2}\right) \qquad 0\leqslant r\leqslant r_2 \tag{33}$$

and the unknowns at a given station in the region become the radius r_2 and the centre-line velocity U_1^*.

The continuity equation, which may be obtained from Eq (11) by replacing U_1 by U_1^* and setting $r_1=0$, becomes:

$$\beta_{30}(U_1^*-U_2)r_2^2+\alpha_{41}=0 \tag{34}$$

and, making similar substitutions in Eq (12), the momentum integral equation taken over the range $(0, r_2)$ gives:

$$\{2\beta_{30}U_2+\beta_{31}(U_1^*-U_2)\}(U_1^*-U_2)r_2^2+\alpha_{42}=0 \tag{35}$$

Eqs (34) and (35) have the solutions:

$$U_1^*=U_2+\frac{\beta_{30}}{\beta_{31}}\left(\frac{\alpha_{42}}{\alpha_{41}}-2U_2\right) \tag{36}$$

$$r_2=-\frac{\alpha_{41}}{\beta_{30}}\left(\frac{\beta_{31}}{2\alpha_{41}U_2-\alpha_{42}}\right)^{1/2} \tag{37}$$

these solutions remaining valid whilst $r_2\leqslant a$.

This analysis, thefore, permits the determination of the radii and velocities at any station in the first two Regions using the equations appropriate to the flow type for any particular pair of values of a and V under consideration.

Region limits

Region I: Region IIA

The flow progresses from Region I into Region IIA when, with r_1 still positive, r_2 equals a. At this juncture Eqs (18) and (19) become:

$$\beta_{11}r_1^2+\beta_{21}ar_1+\beta_{51}a^2=0 \tag{38}$$

$$\beta_{12}r_1^2+\beta_{22}ar_1+\beta_{52}a^2=0 \tag{39}$$

where

$$\beta_{51}=\beta_{31}+a^{-2}\beta_{41}, \qquad \beta_{52}=\beta_{32}+a^{-2}\beta_{42} \tag{40a, b}$$

The condition for Eqs (38) and (39) to possess a common root is:

$$\delta_{15}^2=\delta_{12}\delta_{25} \tag{41}$$

and this leads to a relation which may be written as a quadratic equation in a^2:

$$(\delta_{16}^2+\delta_{12}\delta_{62})a^4+(2\delta_{16}\delta_{17}+\delta_{21}\delta_{27})a^2+\delta_{17}^2=0 \tag{42}$$

where

$$\beta_{61}=\beta_{31}-\tfrac{1}{2}(V-U_2)(V-3U_2)/(U_1-U_2)^2, \quad (43a)$$

$$\beta_{62}=\beta_{32}+\tfrac{1}{2}(V-U_2)(V-4U_1-U_2)/(U_1-U_2)^2 \quad (43b)$$

$$\beta_{71}=-(1-V)(1+V-2U_2)/(U_1-U_2)^2, \quad (43c)$$

$$\beta_{72}=(1-V)(1+V-2U_1)/(U_1-U_2)^2 \quad (43d)$$

Regarding, therefore, the duct radius as unknown, the appropriate solutions are:

$$a=\left[-\left\{\frac{\tfrac{1}{2}\delta_{21}\delta_{27}+\delta_{16}\delta_{17}+|\delta_{21}|(\tfrac{1}{4}\delta_{27}^2-\delta_{17}\delta_{67})^{1/2}}{\delta_{16}^2+\delta_{21}\delta_{26}}\right\}\right]^{1/2} \quad (44)$$

and

$$r_1=\beta_{11}^{-1}\{-\tfrac{1}{2}\beta_{21}+(\tfrac{1}{4}\beta_{21}^2-\beta_{11}\beta_{51})^{1/2}\}a \quad (45)$$

where a is given by Eq (44), and also:

$$a=\left[-\left\{\frac{\tfrac{1}{2}\delta_{21}\delta_{27}+\delta_{16}\delta_{17}-|\delta_{21}|(\tfrac{1}{4}\delta_{27}^2-\delta_{17}\delta_{67})^{1/2}}{\delta_{16}^2+\delta_{21}\delta_{26}}\right\}\right]^{1/2} \quad (46)$$

for those instances for which, with this value of a, the relation in Eq (45) gives a positive value for r_1.

Region I: Region IIB

The condition for passing from Region I into Region IIB rather than into Region IIA is that r_1 decreases to 0 before r_2 attains the value a. Eqs (18) and (19) become:

$$\beta_{31}r_2^2+\beta_{41}=0 \quad (47)$$

$$\beta_{32}r_2^2+\beta_{42}=0 \quad (48)$$

It is necessary, therefore, that:

$$\delta_{34}=0 \quad (49)$$

This leads to:

$$a=\left[-\left\{\frac{2(1-V)\{\beta_{31}(1+V-2U_1)+\beta_{32}(1+V-2U_2)\}}{(V-U_2)\{\beta_{31}(V-4U_1-U_2)+\beta_{32}(V-3U_2)\}}\right\}\right]^{1/2} \quad (50)$$

and this solution is valid provided that the associated value of r_2:

$$r_2=\frac{\left\{\dfrac{\tfrac{1}{2}a^2(V-U_2)(V-3U_2)+(1-V)(1+V-2U_2)}{\beta_{31}}\right\}^{1/2}}{(U_1-U_2)} \quad (51)$$

does not exceed a.

Region IIA: Region IIIA

For the A-configuration the flow leaves the second region when the potential core ceases, i.e. $r_1=0$. This occurs when the fourth term in Eq (30) becomes zero, and this leads to the following quadratic equation for a^2:

$$\begin{aligned}&\{(1-\beta_{30})^2(\tfrac{1}{2}-V^2+2VU_1-\tfrac{3}{2}U_1^2)+(1-\beta_{32})\times(V-U_1)^2\}a^4\\&-2(1-V)\{(1-\beta_{30})^2(\tfrac{1}{2}+\tfrac{1}{2}V-U_1)\\&-(1-\beta_{32})(V-U_1)\}a^2+(1-\beta_{32})(1-V)^2\\&=0\end{aligned} \quad (52)$$

with U_2^* then given by:

$$U_2^*=U_1+\frac{1-V+a^2(V-U_1)}{a^2(1-\beta_{30})} \quad (53)$$

The quartic equation (Eq (52)) produces two positive values of a. The smaller of these, which arises from taking the negative value of the root of the discriminant, is always valid; also required are those solutions obtained from using the positive root which make the associated value of U_2^* in Eq (53) exceed $(V^2+U_1^2-1)^{1/2}$, the value of the notional U_2.

Region IIB: Region IIIB

The third region of the B-configuration is attained when the peripheral potential flow has ceased due to the mixing region reaching the wall. Putting $r_2=a$ in Eq (35) leads to another quadratic equation in a^2:

$$\begin{aligned}&(V-U_2)\{\beta_{30}^2(V-3U_2)-2\beta_{31}(V-U_2)\}a^4\\&+2(1-V)\{\beta_{30}^2(1+V-2U_2)-2\beta_{31}(V-U_2)\}a^2\\&-2\beta_{31}(1-V)^2=0\end{aligned} \quad (54)$$

The requisite solution is associated with the negative square root of the discriminant in Eq (54) and from the continuity equation:

$$U_1^*=U_2+\beta_{30}^{-1}\{V-U_2+a^{-2}(1-V)\} \quad (55)$$

A: B border

For particular values of the radius and velocity ratios a and V there will occur the borderline situation in which the potential primary and secondary flows terminate at a common section. For this condition $r_1=0$ and $r_2=a$, simultaneously, so that Eqs (11) and (12) become:

$$\alpha_{31}a^2+\alpha_{41}=0 \quad (56)$$

$$\alpha_{32}a^2+\alpha_{42}=0 \quad (57)$$

The elimination of a^2 then gives the following quartic relation between W and V:

	V^4	V^3	V^2	V^1	V^0
W^4	$\beta_{13}^2-\beta_{03}^2$	$2\beta_{13}-2\beta_{03}^2$	$2\beta_{13}\beta_{34}+1$	$2\beta_{03}^2+2\beta_{34}$	$\beta_{03}^2+\beta_{34}^2$
W^3	$-2\beta_{03}\beta_{30}+2\beta_{13}\beta_{33}$	$-4\beta_{03}\beta_{30}+2\beta_{33}$	$-2\beta_{13}\beta_{33}+2\beta_{33}\beta_{34}$	$4\beta_{03}\beta_{30}-2\beta_{33}$	$2\beta_{03}\beta_{30}-2\beta_{33}\beta_{34}$
W^2	$\beta_{03}^2-\beta_{30}^2+\beta_{33}^2-2\beta_{13}\beta_{43}$	$2\beta_{03}^2-2\beta_{30}^2-2\beta_{13}-2\beta_{43}$	$-2\beta_{33}^2-2\beta_{13}\beta_{31}-2\beta_{34}\beta_{43}-2$	$2\beta_{30}^2-2\beta_{03}^2-2\beta_{31}-2\beta_{34}$	$\beta_{30}^2-\beta_{03}^2+\beta_{33}^2-2\beta_{31}\beta_{34}$
W^1	$2\beta_{30}\beta_{03}-2\beta_{33}\beta_{43}$	$4\beta_{30}\beta_{03}-2\beta_{33}$	$-2\beta_{31}\beta_{33}+2\beta_{33}\beta_{43}$	$-4\beta_{30}\beta_{03}+2\beta_{33}$	$-2\beta_{30}\beta_{03}+2\beta_{31}\beta_{33}$
W^0	$\beta_{30}^2+\beta_{43}^2$	$2\beta_{30}^2+2\beta_{43}$	$2\beta_{31}\beta_{43}+1$	$2\beta_{31}-2\beta_{30}^2$	$\beta_{31}^2-\beta_{30}^2$

$=0 \quad (58)$

where, additionally to Eq (15):

$$\beta_{03}=1-2\gamma_{11} \qquad \beta_{13}=1-4\gamma_{11}+2\gamma_{12} \qquad (59\text{a, b})$$

$$\beta_{33}=4\gamma_{11}-4\gamma_{12} \qquad \beta_{34}=-\tfrac{1}{2}+4\gamma_{11}-2\gamma_{12} \qquad (59\text{c, d})$$

$$\beta_{43}=\tfrac{1}{2}-2\gamma_{12} \qquad (59\text{e})$$

With the positive value of W appropriate to the given value of V then:

$$a=[(1-V)/\{\beta_{30}(U_1-U_2)-(V-U_2)\}]^{1/2} \qquad (60)$$

where U_1 and U_2 are related to V and W by Eq (5a, b).

The locus of V against a for W lying between 0 and 1 is ploted in Fig 2, the crosses thereon signifying successive changes of 0.1 in the value of W. Also shown are the curves which divide the domain into its respective regions, calculated from the previous analysis of this section, for the pair of values $W=0.0$ and $W=0.5$. When V and W tend to unity from below it is found that:

$$a=\beta_{30}^{-1/2} \qquad (61)$$

this limit representing the smallest value of a for which flow according to the B-configuration is possible.

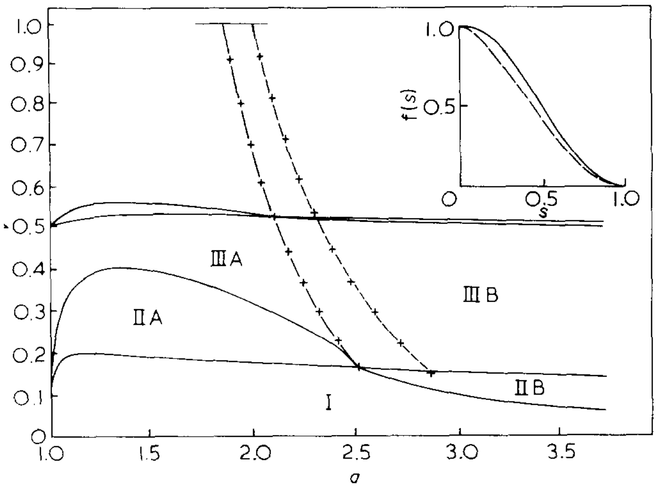

Fig 2 Locus of the vertex for both profiles (shown in inset) and Regions into which the (a, V) plane is divided for W = 0.0 and 0.5. (The identified regions refer to W = 0.0)

The part of this locus which lies above any particular value of V also separates Region IIIA from Region IIIB for the corresponding value of W. As all the Regions I, IIA, IIB, IIIA and IIIB 'meet' at the (a, V) point it is convenient to refer to it as the *vertex* of the flow.

In addition to the locus of the vertex for the trigonometric profile (Eq (17)), the dashed curve shows the corresponding locus for the velocity profile:

$$f(s)=(1-s^{3/2})^2 \qquad (62)$$

which is shown, also dashed, in the inset to Fig 2. The general question of velocity profile is considered in the next section.

Results of the analysis covered by this and the preceding Section are shown further in Figs 3 and 4, which present computations for a convenient value of W, i.e. $W=0.1$. The fixing of W means that, for the appropriate regions of validity in the figures, U_1 and U_2, (which are readily obtained from Eqs (5a, b)), are constant along lines parallel to the a-axis. Fig 3 illustrates r_1 for Regions I and IIA, respectively, calculated from Eqs (24) and (30), and U_1^* for Region IIB, obtained from Eq (36). In Fig 4 are the results for r_2 for Regions I and IIB, found from Eqs (23) and (37), and U_2^* for Region IIA using Eq (32). The cosine profile has been used throughout.

Further considerations

Velocity profile

This problem does not possess a similarity solution, and it is accordingly necessary to choose a velocity profile which seems to complement the experimental data. Mikhail[10] showed that the trigonometric profile (Eq (17)) fitted his results and it is apparent in the literature that it has been a popular choice whenever a mixing region of finite width was under consideration. Squire and Trouncer[9] chose this profile for their analysis of free jets but also considered the alternative profile given by Eq (62).

It is clearly straightforward to evaluate $\{\gamma_{mn}\}$ for any chosen profile and to substitute it in the preceding general analysis. In the present case the adoption of the cosine profile permits direct comparison to be made with the results of Mikhail, see

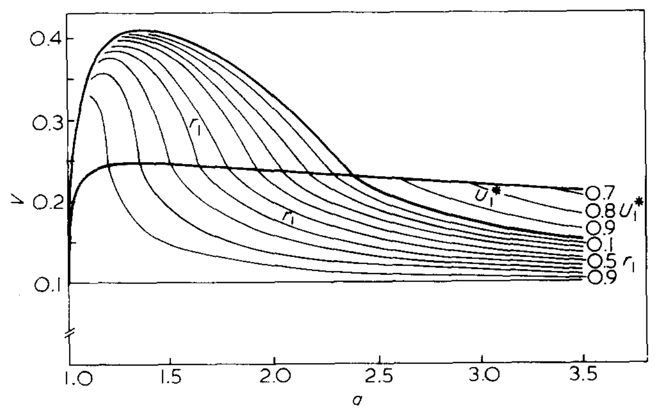

Fig 3 Values of r_1 and U_1^* for W = 0.1

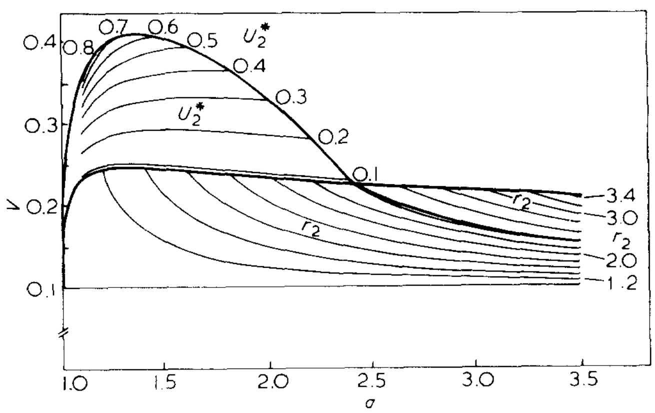

Fig 4 Values of r_2 and U_2^* for W = 0.1

later, and use of the alternative profile allows assessment of the sensitivity of the quantitative results to the profile chosen.

Validity of formulation

The equations have been formulated, as is customary, under the assumptions that the approximations of boundary-layer theory are applicable to the mixing region with potential flow elsewhere, and the nature of these assumptions must always be considered when examining the formal results. There is nothing, for example, in the actual calculation for Region IIA to prevent a negative value appearing for U_2^*. However such a result would indicate the presence of a zone of recirculation, the streamlines would no longer only make small angles with the flow axis and the additional terms necessary in the original equations would then vitiate the ensuing analysis. In particular, results which correspond to very small values of W are not to be relied upon, and any negative value presented for a velocity should merely be regarded as the outcome of formal evaluation for comparative purposes.

Solution uniqueness

It is necessary to establish whether there are any (physical) situations for which more than one solution of the continuity and momentum integral equations is admissible. This is important because, as shown by Fig 2, the flow type for the given geometry, (which is represented by the value of a), may depend upon operating condition, (the value of V), and any hysteretic effect could account for flow instabilities which have been observed, (Middleton[11], Fig 12b).

Any flow of the present type must possess a Region I, the (maximum) extent of which, for a given value of W, is shown in Fig 3 or 4. However, by taking the positive square root of the discriminant in Eq (54), the curve on which $r_2 = a$ and which separates Region IIB from Region IIIB can be extended from the vertex down to the line $V = W$ at the point $a = \beta_{30}^{-1}\beta_{31}^{1/2}$. Similarly, by taking the remaining positive roots of Eq (52) the line on which $r_1 = 0$, separating Region IIA from Region IIIA, can be extended down to the boundary line $V = W$ at the point $a = [(1-\beta_{32})/\{(1-\beta_{32})-(1-\beta_{30})^2\}]^{1/2}$. Therefore, there is a convex triangular region wherein three solutions appear possible, with regions adjacent where two solutions might exist.

On investigation it is found that solution of the Region IIA equations in the extended domain always produce a value of U_2^* less than U_2 whilst solution of the Region IIB equations in the extended domain produces a value of U_1^* exceeding the potential velocity U_1. A comparison of formal solutions for a case which lies within the 'triangle', i.e. $a = 2.4$, $V = 0.2$, $W = 0.1$, is given in Table 2.

Variation with distance downstream

Region I

Although quantities such as r_1, r_2, U_1, U_2, U_1^*, U_2^* are functions of x, the dependence on downstream distance has not yet been explicitly determined. Here, therefore, the procedure of Mikhail is used whereby the momentum-integral equation is applied across a section extending from the centre-line of the duct to the point half-way across the mixing layer. Then, for Region I:

$$u(r_*)\frac{\partial}{\partial x}\int_0^{r_*}\rho r u\,\mathrm{d}r - \frac{\partial}{\partial x}\int_0^{r_*}\rho r u^2\,\mathrm{d}r - \tfrac{1}{2}r_*^2\frac{\mathrm{d}p}{\mathrm{d}x}$$
$$= -\tau(r_*)r_* \tag{63}$$

where

$$r_* = \tfrac{1}{2}(r_1 + r_2) \tag{64}$$

Adopting Prandtl's momentum transfer theory, Mikhail wrote:

$$\tau(r) = \rho c_I^2(r_2 - r_1)^2\left(\frac{\partial u}{\partial r}\right)^2 \tag{65}$$

where c_I is an empirical constant.

Eq (63) then becomes:

$$\{U_2 + (U_1 - U_2)f(\tfrac{1}{2})\}\frac{\mathrm{d}}{\mathrm{d}x}\Big[\tfrac{1}{2}r_1^2U_1 + \int_0^{1/2}\{r_1 + (r_2 - r_1)s\}$$
$$\times\{U_2 + (U_1 - U_2)\mathrm{f}(s)\}(r_2 - r_1)\,\mathrm{d}s\Big]$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big[\tfrac{1}{2}r_1^2U_1^2 + \int_0^{1/2}\{r_1 + (r_2 - r_1)s\}$$
$$\times\{U_2^2 + 2(U_1 - U_2)U_2\mathrm{f}(s) + (U_1 - U_2)^2\mathrm{f}^2(s)\}$$
$$\times(r_2 - r_1)\,\mathrm{d}s\Big] + \tfrac{1}{8}(r_1 + r_2)^2U_2\frac{\mathrm{d}U_2}{\mathrm{d}x}$$
$$= -\tfrac{1}{2}c_I^2(r_1 + r_2)(U_1 - U_2)^2\{\mathrm{f}'(\tfrac{1}{2})\}^2 \tag{66}$$

Investigating this equation further it is found that it assumes the form:

$$g_1(r_1, r_2)U_1\frac{\mathrm{d}U_1}{\mathrm{d}x} + g_2(r_1, r_2)U_2\frac{\mathrm{d}U_2}{\mathrm{d}x}$$
$$+ g_3(r_1, r_2)U_1\frac{\mathrm{d}U_2}{\mathrm{d}x} + g_4(r_1, r_2)U_2\frac{\mathrm{d}U_1}{\mathrm{d}x}$$
$$+ h_1(U_1, U_2)r_1\frac{\mathrm{d}r_1}{\mathrm{d}x} + h_2(U_1, U_2)r_2\frac{\mathrm{d}r_2}{\mathrm{d}x}$$
$$+ h_3(U_1, U_2)\left\{r_1\frac{\mathrm{d}r_2}{\mathrm{d}x} + r_2\frac{\mathrm{d}r_1}{\mathrm{d}x}\right\}$$
$$= \text{an algebraic function of } (r_1, r_2, U_1, U_2) \tag{67}$$

where the $\{g_i\}$ are functions which do not have explicit dependence on U_1, U_2, and the $\{h_i\}$ are functions which are similarly independent of r_1, r_2.

Table 2 Comparison of formal solutions for the case $a = 2.4$, $V = 0.2$, $W = 0.1$

Region	U_1	U_1^*	r_1	r_2	U_2	U_2^*
I	0.985	—	0.193	2.156	0.098	—
IIA	0.985	—	0.060	—	—	0.052
IIB	—	1.063	—	2.197	0.098	—

By Eqs (1) and (2) the quantities $\{U_i(\mathrm{d}U_j/\mathrm{d}x),\ i, j = 1, 2\}$ can be expressed as functions of U_2 and $\mathrm{d}U_2/\mathrm{d}x$. To obtain as functions of U_2 and $\mathrm{d}U_2/\mathrm{d}x$ expressions for $r_1(\mathrm{d}r_1/\mathrm{d}x)$ and $r_2(\mathrm{d}r_2/\mathrm{d}x)$ the squares of Eqs (23) and (24), respectively, are differentiated. Finally, to handle $(r_1(\mathrm{d}r_2/\mathrm{d}x)+r_2(\mathrm{d}r_1/\mathrm{d}x))$, Eq (18) is differentiated and then the preceding results are used. It is thus possible to write Eq (67) in the form:

$$G_1(U_2)\frac{\mathrm{d}U_2}{\mathrm{d}x} = c_{\mathrm{I}}^2 G_2(U_2) \tag{68}$$

where G_1, G_2 are functions only of U_2 and not its derivative. Therefore:

$$c_{\mathrm{I}}^2 x + \text{constant} = \int \frac{G_1(U_2)}{G_2(U_2)}\,\mathrm{d}U_2 \tag{69}$$

and this integration may be carried out numerically. The constant is such that $x = 0$ when $U_2 = U_s$.

Region IIB

It is possible to proceed in similar manner for Region IIB. Assuming that the turbulent shearing stress may be represented by:

$$\tau(r) = \rho c_{\mathrm{IIB}}^2 r_2^2 \left(\frac{\partial u}{\partial r}\right)^2 \tag{70}$$

where c_{IIB} is a second empirical constant, it is possible, analogously to Eq (66), to obtain:

$$\begin{aligned}
&\{U_2+(U_1^* - U_2)\mathrm{f}(\tfrac{1}{2})\}\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^{1/2} r_2^2 s\{U_2 + (U_1^* - U_2)\mathrm{f}(s)\}\,\mathrm{d}s\right] \\
&-\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^{1/2} r_2^2 s\{U_2^2 + 2U_2(U_1^* - U_2)\mathrm{f}(s) + (U_1^* - U_2)^2 f^2(s)\}\,\mathrm{d}s\right] \\
&+\tfrac{1}{8} r_2^2 U_2 \frac{\mathrm{d}U_2}{\mathrm{d}x} = -\tfrac{1}{2} c_{\mathrm{IIB}}^2 r_2 (U_1^* - U_2)^2 \{\mathrm{f}'(\tfrac{1}{2})\}^2
\end{aligned} \tag{71}$$

To express this equation in terms of U_2 and $\mathrm{d}U_2/\mathrm{d}x$, it is then necessary to substitute for terms such as $\mathrm{d}\{r_2^2(U_1^* - U_2)\}/\mathrm{d}x$ and this can be accomplished by differentiating the continuity equation, Eq (34). Then, using Eq (36) and (37) for U_1^* and r_2 it is again possible to obtain an ordinary differential equation, similar to Eq (68) but involving two new functions $G_3(U_2)$ and $G_4(U_2)$, the solution of which may be written:

$$c_{\mathrm{IIB}}^2 x + \text{constant} = \int \frac{G_3(U_2)}{G_4(U_2)}\,\mathrm{d}U_2 \tag{72}$$

The constant is such that this solution matches the solution to Region I at their interface.

Region IIA

Under the assumption that, with third empirical constant c_{IIA},

$$\tau(r) = \rho c_{\mathrm{IIA}}^2 (a - r_1)^2 \left(\frac{\partial u}{\partial r}\right)^2 \tag{73}$$

the momentum integral equation again taken halfway across the mixing region is obtained:

$$\begin{aligned}
&\{U_2^* + (U_1 - U_2^*)f(\tfrac{1}{2})\}\frac{\mathrm{d}}{\mathrm{d}x}\left[\tfrac{1}{2}r_1^2 U_1 + \int_0^{1/2} \{r_1 + (a - r_1)s\} \times \{U_2^* + (U_1 - U_2^*)\mathrm{f}(s)\}(a - r_1)\,\mathrm{d}s\right] \\
&-\frac{\mathrm{d}}{\mathrm{d}x}\left[\tfrac{1}{2}r_1^2 U_1^2 + \int_0^{1/2} \{r_1 + (a - r_1)s\} \times \{U_2^{*2} + 2U_2^*(U_1 - U_2^*)\mathrm{f}(s) + (U_1 - U_2^*)^2\mathrm{f}^2(s)\}(a - r_1)\,\mathrm{d}s\right] \\
&+\tfrac{1}{8}(a + r_1)^2 U_1 \frac{\mathrm{d}U_1}{\mathrm{d}x} \\
&= -\tfrac{1}{2}c_{\mathrm{IIA}}^2 (a + r_1)(U_1 - U_2^*)^2\{\mathrm{f}'(\tfrac{1}{2})\}^2
\end{aligned} \tag{74}$$

Again it is possible to make recourse to the appropriate continuity and momentum equations, and to their derivatives, but the more complicated equation for r_1 makes the manipulation even heavier.

Comparisons and conclusions

Comparison with Mikhail

Mikhail presented (most of) his results for the set of theoretical cases which arose from considering three values of V, i.e. 0.1333, 0.2 and 0.4, with each value of a, i.e. 2.0, 3.333 and 10.0. The first nine lines of Table 3 contain the comparison between his calcula-

Table 3 Comparison of the results of Mikhail and Middleton

a	V	Termination of Region I								Termination of Region II					
		$c_{\mathrm{I}}^2 \times$ length		U_1		U_2		r_2		$c_{\mathrm{IIB}}^2 \times$ length		U_1^*		U_2	
		Mik	Mid	Mik	Mid	Mik	Mid	Mik	Mid	Mik	Mid	Mik	Mid	Mik	Mid
2.000	0.133	0.068 0.058	—	1.00	—	0.133	—	1.50	—	0.018 0.028	—	0.87	—	0.133	—
	0.200	0.072 0.055	—		—	0.200	—	1.50	—	0.022 0.035	—	0.89	—	0.200	—
	0.400	0.10 0.070	—		—	0.400	—	1.42	—	0.028 0.060	—	0.92	—	0.400	—
3.333	0.133	0.068 0.058	0.049		0.9936	0.133	0.0715	1.83	2.380	0.17 0.19	0.064	0.41	0.7693	0.133	−0.0249
	0.200	0.070 0.062	0.058		0.9915	0.200	0.1521	1.83	2.311	0.23 0.23	0.11	0.46	0.7289	0.200	0.0786
	0.400	0.098 0.098	0.090		0.9916	0.400	0.3785	1.83	2.144	0.35 0.34	0.30	0.57	0.7018	0.400	0.3492
10.000	0.133	0.44	0.051		0.9993	0.133	0.1282	4.00	2.307	?	0.57	?	0.2652	0.133	0.0898
	0.200	0.45	0.058		0.9992	0.200	0.1958	3.76	2.256	?	0.79	?	0.2922	0.200	0.1724
	0.400	0.45	0.086		0.9992	0.400	0.3979	3.16	2.125	?	1.4	?	0.4402	0.400	0.3915
3.846	0.200	0.073	0.058		0.9939	0.200	0.1667	?	2.292	0.31	0.17	0.38	0.6349	0.200	0.0929

tions and the present analysis. Unfortunately, his diagrams do not appear to be completely consistent and where two values for ($c^2 \times$length) appear in Table 3 the upper one refers to the value indicated in his Fig 3a or Fig 3b, and the lower one to that obtained from his Fig 4. It is also difficult to understand why, on the basis of his analysis, using his Eq (9), he does not obtain the value of r_2 at the end of Region I to always equal 1.83. The final line in Table 3 pertains to the case $a = 3.846$, $V = 0.2$ which Mikhail was able to investigate experimentally and the results for which appear in his Fig 10. Understandably, he found the practical determination of r_2 to be too difficult.

It is apparent that there are serious discrepancies between the results here and those of Mikhail, notwithstanding that Mikhail commenced with the same equations and used the same (cosine) velocity profile. He restricted himself, however, to consideration of the B-configuration, whereas Fig 2 confirms that for $a = 2.0$ all three velocity ratios give rise to A-configurations so that for this smallest value of a further comparison of results would be specious.

In addition to pronounced differences for the velocities and for the radii between the two sets of results, there appear large differences in the values of the mixing lengths which, in Table 3, have been given to two significant figures. As the c^2 values are arbitrary in the sense that they may be chosen to give the best agreement between theory and experiment, the difference between the values of Mikhail and those here are only of significance in that, because of the manner of their computation, there should be agreement. In Fig 5 the present results and those of Mikhail, (taken from his Fig 3a, Fig 4 and with r_1 computed from his Eq (9)), are compared for the case $a = 3.333$, $V = 0.4$ after the lengths of Region I and Region II have all been normalised to unity. Any change of slope at the interface in this Figure has, therefore, no significance here.

That the results of Mikhail are at such variance with the values here which have been obtained as exact solutions of the same starting equations stem from his attempts at linearisation. In the notation of this paper he argued that 'the variation in the value of U_1 with x is negligible ... [and that] ... experimental evidence validated the assumption of constant U_2 in both the first [Region I] and second [Region II] mixing regions'. It is certainly true, as Table 3 and Fig 5 confirm, that the decay of U_1 is very small. Nevertheless, concern in the momentum integral equation is rather with terms such as

$$U_1 \frac{dU_1}{dx}, \qquad U_2 \frac{dU_2}{dx}$$

and the pressure gradient and these are all of the same order of magnitude, being specifically related by Eq (1). To allow for *none* of these gradients produces momentum and continuity equations the only solution of which is $U_1 = U_p$, $U_2 = U_s$, $r_1 = r_2 = 1$, indicating that no mixing is taking place. Similarly, merely to allow for *some* of these gradients implies that the approximations made are inconsistent so that spurious results ensue.

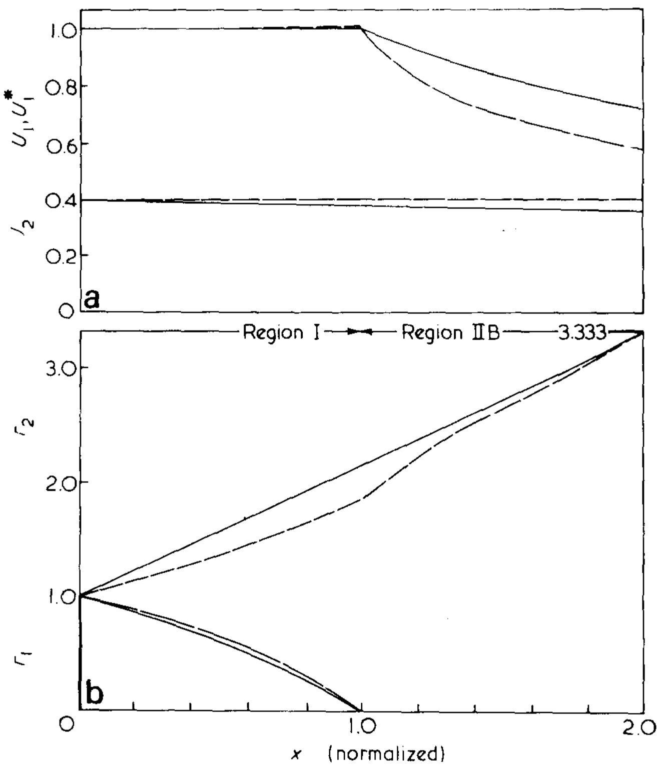

Fig 5 Comparison of the results of Mikhail -- and Middleton —— for a = *3.333,* V = *0.4, with normalized Region lengths. (a) Velocities (b) Radii*

It is, therefore, worth restating that in the absence of a pressure gradient for an unshrouded viscous jet, it is the influx of mass from infinity that allows the mixing to proceed. In the case of the shrouded jet it is the induced pressure gradient which is essential for the mixing. For these reasons the use of a global continuity equation by Craven[12] was incorrect in his work on density effects in the mixing of free jets, as is also the assertion by Duggins[13] that the analytic results for a free jet will correspond to those of a ducted jet as the radius of the duct tends to infinity.

Conclusions

In this analysis of the simple model, the regions of flow have been identified, their boundaries established and values for the radii of the streams and the velocities at any section from the discharge plane up to the disappearance of potential flow determined. Some contour graphs have been given for a cosine velocity profile but any suitable profile may be used. Such general graphs could be used to aid design or alternatively the flow patterns for a specified pair of values for a and V could be evaluated. For the explicit determination of the limits of the flow regions it is necessary to treat a as an unknown, so that for a specified value of duct radius such limits could only be found by iteration. Nevertheless, for given duct size the flow field may be established merely by ensuring that the formal solutions consist of values which are physically acceptable.

Subject to the stated assumptions, these solutions are exact so that attempts at linearisation of the equations are unnecessary and may lead to results which are grossly in error.

Apart from failure to handle situations such as the zones of recirculation which may exist, where the assumptions of the model are clearly violated, the weakest part of the analysis is that which describes the development of the flow with distance downstream. Calculations which use a mixing length theory to model the turbulence generally do provide values which broadly reflect the (time-invariant) situation. In part this is due to the flexibility of the empirical constant c. To choose, for example, a point other than half-way across the mixing region as the upper limit of integration would produce a very similar picture of the flow development but with a different value accordable to c for best agreement between the theory and practical results. The calculations for this part are sufficiently heavy and of sufficient dubiety that it would seem expedient to find, if possible, either as a result of the analysis or by experiment, a parameter which varies in simple manner, and over considerable ranges of a and V, with the distance downstream, and relate the other parameters thereby. Inspection of the computations carried out shows that in Region I the increase in the outer radius r_2 is effectively rectilinear with x. Agreement is not so good for Region IIB but nevertheless r_2 remains the best choice as shown in Fig 5.

These results constitute a sound framework from which it is possible to make extensions, though possibly on a more piecemeal basis, to take account of further effects. These include the mixing of fluids of differing densities, the presence of wall boundary layers and the nozzle wake, flaring of the duct, and entry effects. The computations can be extended to the case where the secondary velocity exceeds that of the primary jet, $(V>1)$, and may be re-worked for the corresponding two-dimensional (plane) situation.

Appendix

Choice of dimensionless quantities

Comparison of the work of various researchers is made difficult by the large number of parameters involved. Although it is clearly desirable to produce some standardisation the most appropriate quantities to use for non-dimensionalisation are not self-obvious.

Length

Length does not pose too great a problem. In the axi-symmetric case, the geometrical quantities which involve distance are the radii (or diameters) of the primary nozzle and duct, (together with their squares and corresponding reciprocals), with the distance downstream and radius of the mixing region entering but indirectly.

If the procedure of Mikhail were used all distances would be in terms of duct radius and the primary jet would then be issuing from a nozzle with a dimensionless radius which lay between 0 and 1. However, it may then be found that if plotted on a rectilinear scale the resulting graphs become congested. As it is the *duct* which can generally be altered if any physical change to the system is possible, it appears reasonable, therefore, to express all lengths in terms of the radius of the primary nozzle.

Velocity

Whichever measure is chosen for unit length the transformation from that to another is decidedly trivial, but the same is not true for the velocities as the four choices U_p, U_s, U_1, U_2 are linked non-rectilinearly by Eq (1). The frequent occurrence of U_1 in the equations means that for certain calculations this could be an appropriate choice, but due to the dependence of U_1 on x and its sometimes notional existence the most suitable unit appears to be U_p, the velocity of the jet at its efflux plane. (Mikhail did not, in effect, distinguish between U_p and U_1).

Thus, the equations and results have been presented accordingly.

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